Beginning Lecture Course: Week 2
"Knots and their invariants"

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notes by Ellen Goldstein

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<u>Def^n</u>: A <u>knot</u> is a smooth embedding of S^1 into S^3. Equivalently, one may think of it as a polygonal simple closed curve in S^3 or $\mathbb{R}^3$.

<u>Non-Example:</u>

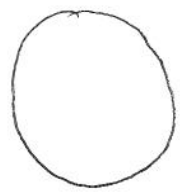
An infinite chain of shrinking knots

It is sometimes useful and completely accurate to use a shoelace or other string joined end to end to represent a knot. This leads to a natural definition of when two knots are equivalent:

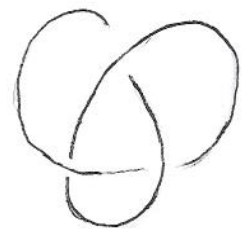
<u>Def^n</u>: Two knots, K^1 and K^2, are equivalent, $K^1 \cong K^2$, if there is a smooth deformation of K^1 onto K^2 such that the knot does not

intersect itself.

These two knots are not equivalent



the unknot
or trivial knot



a trefoil
knot

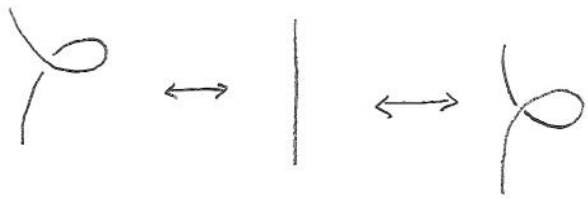
since the trefoil cannot be continuously deformed in such a way that it no longer has any crossings.

The above pictures are called <u>knot diagrams</u> and are projections of the three-dimensional knot into $\mathbb{R}^2$ (or S^2) that include crossing information. An undercrossing is represented as a break in the curve.

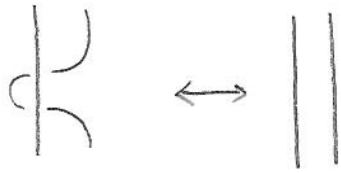
Clearly, two knots that are equivalent should have equivalent diagrams, so we define a way to transform the diagrams without changing the knot depicted. These are known as Reidemeister moves.

Reidemeister moves

Type I:



Type II:



Type III:

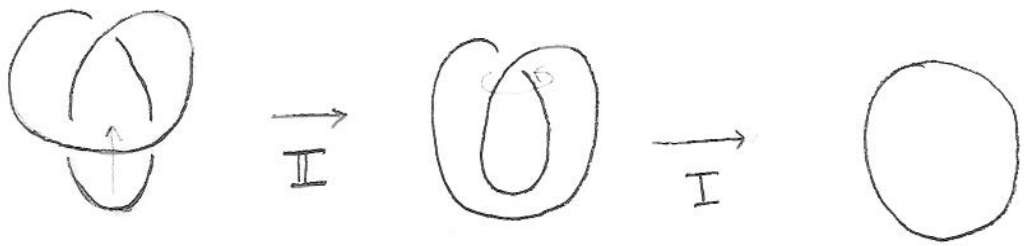
Note: A type 3 move may be thought of as moving any of the three strands past the intersection of the other two.

Theorem (Reidemeister 1927): Two diagrams represent the same knot if and only if they are related by a finite sequence of Reidemeister moves.

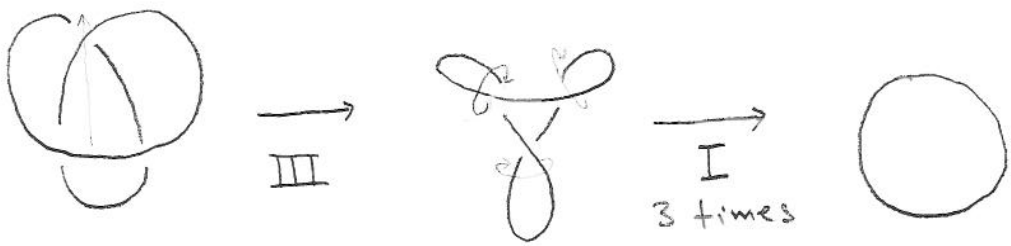
Proof omitted/left to reader.

<u>Example</u>:

The following knot is equivalent to the unknot, demonstrated by Reidemeister moves on the knot diagram:



<u>or</u>

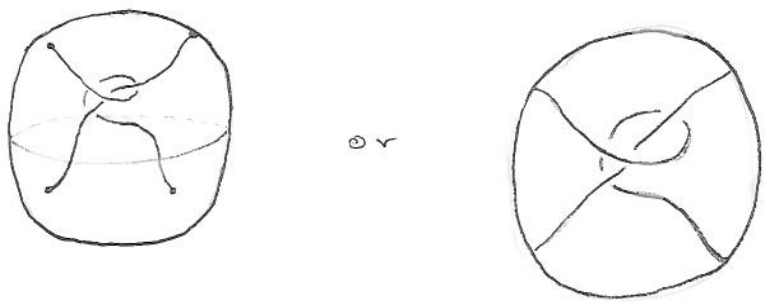


Given two knot diagrams, it is often a nontrivial problem to find whether there is a series of Reidemeister moves from one to another. Generally, one uses other knot invariants to distinguish two knots. Any invariant property of a knot must be left unchanged by Reidemeister moves on the knot diagram.

Building knots

A. Conway (1970)

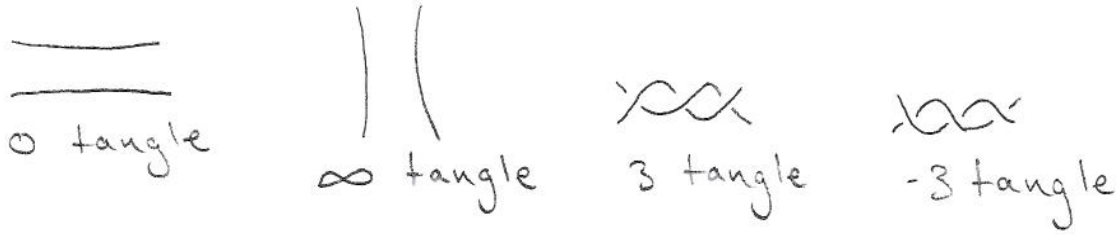
John Conway introduced the concept and standard notation for a tangle, which, for our purposes, is two arcs embedded in a ball (or disc) such that each begins and ends on the boundary:



Tangles are equivalent if you can change one into the other without

- intersecting the tangle,
- moving the endpoints, or
- leaving the ball.

Examples:



Note: One generally omits the disc or ball.

An n-tangle has n crossings where the over-crossing has positive slope (a left-handed crossing), while a -n-tangle has n crossings where the overcrossing has negative slope. The tangle is horizontal.

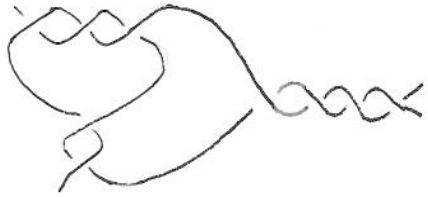
To multiply two tangles, one must first <u>reflect</u> one tangle across a diagonal:

reflection as in a mirror

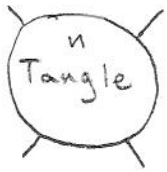
and then add

2 -3 tangle

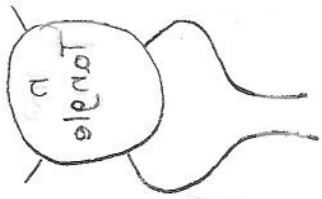
<u>Example</u>: 3 2 -4 tangle



One may begin or end with a vertical tangle by multiplying by a zero tangle.



n tangle



n 0 tangle

To each tangle we attach a rational number:

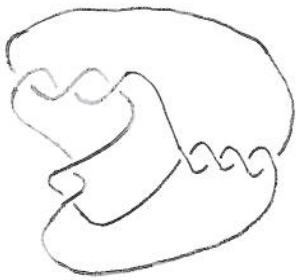
$3\ 2\ -4$ corresponds to $-4 + \dfrac{1}{2 + \frac{1}{3}}$

$x_1\ x_2 \dots x_n$ corresponds to $x_n + \dfrac{1}{x_{n-1} + \dfrac{1}{x_{n-2} + \dfrac{1}{\ddots + \dfrac{1}{x_2 + \frac{1}{x_1}}}}}$

<u>Theorem</u>: Two rational tangles are equivalent if and only if they correspond to the same rational number.

Proof: one direction is an exercise. The other is difficult and omitted.

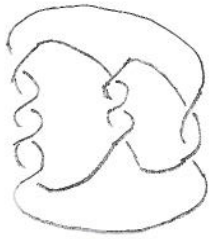
One gets a knot by connecting the top two and bottom two endpoints:



$3\ 2\ -4$ tangle knot

The sum of two tangles, T_1 and T_2, is denoted T_1, T_2:

Prezel knots can be written as sums of tangles:



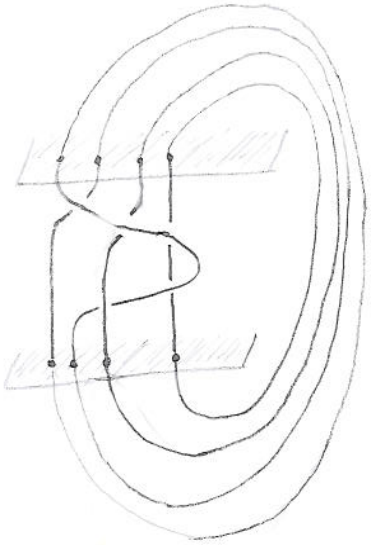
4, 3, 2

though not all knots can be.

B. Braids

<u>Def^n</u>: An <u>n-string</u> <u>braid</u> consists of n arcs that are fixed at their endpoints to a "ceiling" and "floor" such that the strands do not intersect and are monotone decreasing.

One connects top to bottom to make a knot or link:

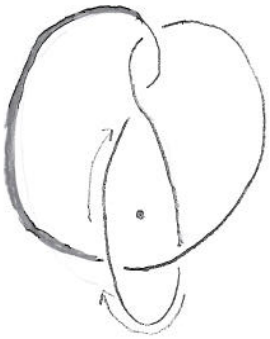


The closure of a 4-string braid.

<u>Theorem</u> (Alexander 1923): Every knot is the closure of a braid.

Example:

Figure 8 knot:



deform the knot so that the arcs circle a chosen point in a chosen direction (in this case, clockwise).

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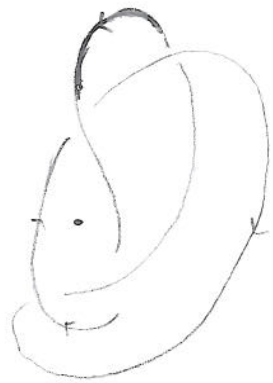

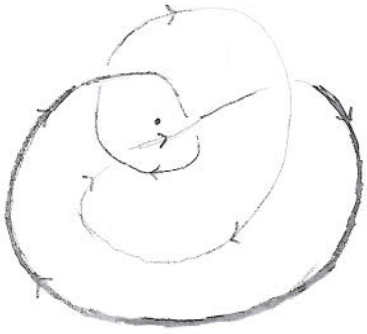

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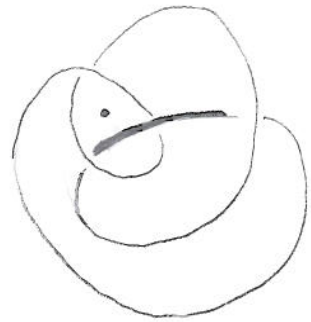

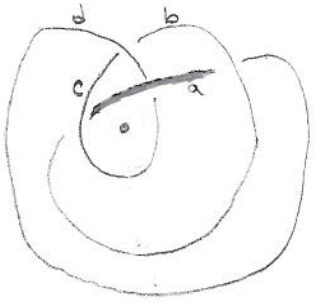

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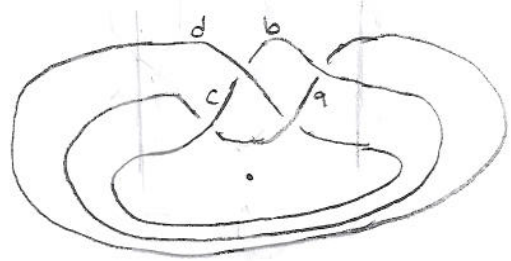
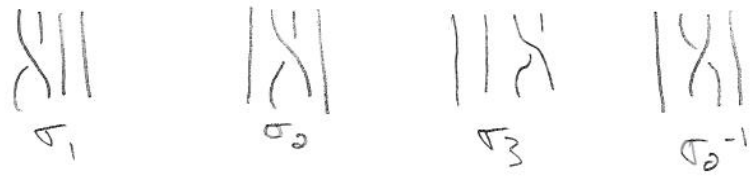


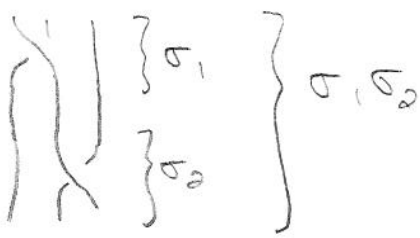
Crossings in braids are labled by the following convention:



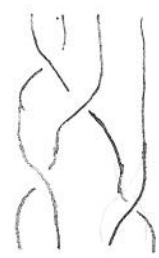
etc

so that σ_i denotes the ith strand crossing over the i+1-strand and σ_i^{-1} denotes the ith strand crossing <u>under</u> the i+1-strand.

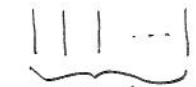
One multiplies braids on n strands by stacking them, thus forming a group:



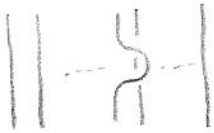
$\sigma_1\sigma_2$



$\sigma_1\sigma_2^{-1}\sigma_1\sigma_4^{-1}$

With identity: 

no twists

Inverses: 

Identity by R.M.II

$\sigma_i\sigma_i^{-1} = 1$

And some other useful properties:

- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

Reidemeister III

<u>Theorem</u> (Markov 1935): The closure of two braids give the same knot if the two braids are related by the following moves:

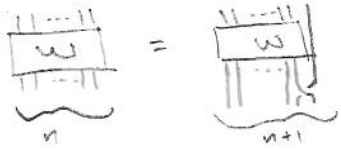
i) $\sigma_i \sigma_i^{-1} = 1$ (RM II)

ii) $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i-j| > 1$

iii) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$

iv) $\sigma_j w \sigma_j^{-1} = w$ where w is any "word" in σ_i's

v) "stabilization"



ie $w = w\sigma_{n+1}$ where w is a word in n strands.

The proofs of (iv) and (v) are left as exercises.