Lecture 3
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These notes are a rush transcript of Rachel's lecture. Any mistakes or typos are mine, and please point these out so that I can post a corrected copy online.

1 Introduction and related material

Recall that we're studying codimension-one foliations of M: decompositions of M into surfaces (often non-compact) with the property that locally, the decomposition looks like a product. Foliations and related techniques have played an important role in low-dimensional topology.

For examples, Reebless foliations give us topological information about our manifold. If $M \neq S^1 \times S^2$ has a Reebless foliation, M is irredicible, $\pi_1(M)$ is infinite, and $\tilde{M} = \mathbb{R}^3$. Also, if $\gamma \pitchfork \mathcal{F}$, then $[\gamma]$ is infinite order in $\pi_1(M)$. (Reuslts of the 1960's)

Furthermore, work of Gabai lets us use Reebless foliations to detect whether a surface has minimal genus in its homology class. If the surface is of minimal genus, it can be extended to be a leaf of a foliation. (Reuslts of the 1980's)

More recently (2003ish), Tau Li proved the generalized Waldhausen Conjecture using normal surface theory and branched surfaces:

Theorem 1. *(Li, Jaco-Rubenstein) Let M be orientable, irreducible, and atoroidal. Then for every g there exist at most finitely many Heegaard decompositions of genus g, up to isotopy.*

In 2007 Heegaard Floer homology and sutured manifold theory were used to prove the following theoreom:

Theorem 2. *(Yi Ni) If $K \subset S^3$ admits a lens space surgery, K is fibered.*

That is, if there is a Dehn surgery on K such that the resulting manifold has a genus-one Heegaard decomposition, then the complement of the original knot was a fibered three manifold.

2 Laminations

Today's topics are laminations, branched surfaces, and sutured manifold theory.

By way of motivation, note that if M is hyperbolic, it "often" contains a Reebless foliation. For example, this is true more often than it's true that they contain essential surfaces.

Recall the examples from yesterday of the form $M = M_0 \cup_\phi (S^1 \times D^2)$. Here $M_0 = F \times [0,1]/(x,0) \sim (fx,1)$ where F is a compact orientable surface with boundary equal to S^1, and f is a homeomorphism of F. When do such manifolds clearly have Reebless foliations?

2.1 Foliations and laminations of surfaces

Begin by considering the analogous situation one dimension lower. Let F be a closed orientable surface with $\mathcal{F}$ a one-dimensional foliation of F. But this is not so interesting....

Fact 1. *F contains a foliation if an only if F has a nowhere-vanishing vector field if an only if $\chi(F) = 0$.*

(See Milnor's *Topology from a differential viewpoint* for a proof of this.)

Instead, let's consider singular foliations.

Definition 1. $\mathcal{F}$ is a *singular foliation* if $\mathcal{F}|_{F\setminus\{p_1,\dots p_n\}}$ is a foliation and in a neighborhood of each p_i, we see a singularity with $n \geq 3$:

Figure 1: Local models for singular foliations at singular points, $n = 3, 4, 5$.

The leaves which aren't manifolds are called *singular leaves*. We can split open the singular leaves to get a *lamination* λ, a foliation of a closed subset of F.

Figure 2: Splitting open a singularity to get a lamination

A lamination has a local atlas of charts of the form $\lambda \cap U = \{\mathbb{R} \times x | x \in \textit{closed subsest of } \mathbb{R}\}$.

2.2 Automorphisms of surfaces

Recall Thurston's classification of automorphisms of surfaces:

Theorem 3. *Let F be a compact orientable surface with $\chi(F) < 0$ and let $f \in Homeo^+(F)$. Then f is isotopic to some homeomorphism g such that*

1. *$g^n = 1$ for some $n \neq 0$; or*
2. *there exists a collection of disjoint, properly embedded curves fixed by g; or*
3. *g has neither of the previous two properties and there exists a pair of transverse laminations λ^s, λ^u fixed by g.*

These three cases are called *periodic*, *reducible*, and *pseudo-Anosov*, respectively. In the construction described above, M_0 is the mapping torus of F under some map f, and classifying the homeomorphism of F using Thurston's theorem tells us about the resulting manifold M . If g is periodic, M_0 is fibered by circles and M is Seifert fibered. In the reducible case there exist essential annuli or tori on M_0. In the pseduo-Anosov case, M admits a constant (-1) curvature metric, so it is hyperbolic.

Definition 2. We call a lamination of F *essential* if the leaves are π_1-injective and there does not exist a monogon complementary region. (See Figure 3)

Figure 3: Some cusped n-gons: monogon, punctured bigon, trigon, four-gon

Example: Let $F = D^2$. An example of a non-essential lamination is suggested in Figure 4.

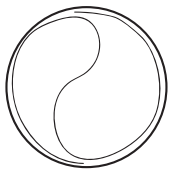

Figure 4: Non-essential lamination of D^2 with two monogon complementary regions.

2.3 Suspending a lamination

We suppose that M_0 is hyperbolic. Letting λ be one of the essential laminations fixed by f, define the *suspension* of λ by $\Lambda = \lambda \times [0,1]/(x,0) \sim (fx,1)$.

If we could extend Λ to a Reebless foliation of M, then $\tilde{M}$ is $\mathbb{R}^3$ and Perelman's result lets us classify the geometry of our manifold. Sometimes this can be done...

The complementary regions of λ are cusped polygons or once-punctured cusped polygons (Figure 3). When the complementary regions of λ are cusped polygons with at least three cusps, their images in the suspension are "spiked" solid tori: i.e, solid tori with simple closed curves removed from the boundary. Punctured complementary regions become spiked solid annuli. We can foliate the four-spiked torus with S^1's worth of the saddle surfaces shown in Figure 5. This "fills up" the complementary regions of Λ with a Reebless foliation.

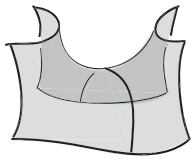

Figure 5: A saddle surface.

Gabai and Oertel showed that *essential* laminations are a good tool for studying three manifolds. In particular, *good laminations* have π_1-injective leaves; no monogon complementary regions; no S^2 leaves; and the complementary regions are incompressible. One reason these are interesting is that Dehn surgeries on tori disjoint from essential laminations with "usually" produce manifolds in which the lamination is still essential.