

We defined the Fukaya category of a symplectic manifold M last time.

Objects were Lagrangian manifolds with extra data (flat unitary bundle). Morphisms were chain complexes spanned by intersection

The product was given by a count of pseudo holomorphic disc.

Example: If our symplectic manifold is T^2 , the torus then every Lagrangians which is not in the constant isotopy class is Hamiltonian isotopic to a unique straight line for some metric. This metric is irrelevant, except that it makes things easier to draw.

Let L_Q be a $(1,0)$ curve, L_P a $(0,1)$ curve, and L_C a $(1,1)$ curve. One can check that there is a non-trivial product in the Floer homology between these three curves. The notation is suggestive of a theorem of Polishuk and Zaslow which proves the homological mirror conjecture of Kontsevich in this case. The correspondence takes Lagrangians to stable sheaves, and maps the homology class to the rank and the degree.

The moduli of elliptic curves is mirror to a choice of B and ω . Explicitly, if $E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ corresponds to forms satisfying

$$\int_{T^2} B + i\omega = \tau.$$

It is true (but hard to check) that the Fukaya category is invariant under the modular transformations.

Another example: Consider $\mathbb{C}P^1$, which is symplectically S^2 , and is determined by its area. Lagrangians are just circles in S^2 , but all Lagrangians which don't separate the sphere into two discs of equal part are displaceable by Hamiltonian isotopy. Since Floer homology is invariant under such isotopies, the Floer homology vanishes.

On the other hand, L , the equator is more interesting. Any Hamiltonian deformation of L still intersect L in at least two points. Assume that we get 2 points p and q . We can check that all holomorphic discs come in pairs of equal area. Going

through the formalism for signs, one sees that these discs cancel, so the Floer differential vanishes, and the Floer homology is the ordinary homology. The product structure is not the ordinary cup product, but rather a deformation into a Clifford Algebra.

Also, we can choose a flat rank one bundle. It turns out that only bundles with holonomy $-\mathcal{I}d$ have non-zero Floer homology.

To understand mirror symmetry for these objects, we consider a Landau-Ginzburg model on $Y = \mathbb{C}^*$ with coordinate z and $W = z + e^{-t}/z$, where t is the area of the circle. Note that the critical points are $\pm ze^{-t/2}$. If we consider the category of singularities for this Landau-Ginzburg model, we get exactly the Fukaya category of the mirror S^2 .

In fact, the moduli ω does not affect the equivalence class of the Fukaya category. However, once we keep track of m_0 , we can recover the equivalence “on the nose.” Indeed, if L is a non-equatorial Lagrangian, then $m_0(L) \in CF^0(L, L)$ is of the form $e^{-A} + e^{-(t-A)}$. The value of the critical point is therefore recovered by m_0 .

To recover the other side of mirror symmetry, we will study the category of coherent sheaves on \mathbb{CP}^1 . Its mirror will be a category of Lagrangians,

Let Y be a non-compact manifold, equipped with a symplectic form ω , and a 2-form B . Consider a superpotential $W: Y \rightarrow \mathbb{C}$ a “superpotential” whose fibers Σ_λ are symplectic submanifolds when they are smooth. For exaple, we can let Y be Kaehler, and W be a holomorphic function.

With this data, we can define a parallel transport between fibers away from the critical points. The symplectic orthogonal to the tangent space of the fiber is a horizontal distribution which can be used to define this transport. There is a problem with infinity, and with singularities. Ideally, the fibers would either be complete or compact, so the problem at infinity would disappear. An easy computation in symplectic topology then implies that parallel transport defines a symplectomorphism between the smooth fibers.

In particular, unlike a complex fibration, there is no “variation of symplectic structures” and all fibers are “the same”. There is still an interesting phenomenon of global monodromy.

Definition 0.1. (*Hori, Vafa, Kontsevich, Seidel, ...*) *A properly embedded Lagrangian submanifold $L \subset Y$ is admissible (with phase 0) if, outside of a compact subset, L is invariant under parallel transport in the negative real direction.*

This definition is equivalent to the condition that, outside of a compact set, the image of L under W agrees with a union of half rays going to $-\infty$ along the real axis. We can generalize this to a notion of admissible Lagrangians with phase α , whose projections (outside a compact set) agrees with a half ray of angle α with the real axis. We can always replace the negative real half-ray by a ray which is asymptotically equal to a half ray of angle α .

Definition 0.2. *The category $\mathcal{F}(Y, W, \omega, B)$ as objects (L, E, ∇) where L is a compact admissible Lagrangian of phase 0, etc.*

We will define $\text{Hom}^(\mathcal{L}, \mathcal{L}') := CF^*(\mathcal{L}(\alpha), \mathcal{L}'(\alpha'))$ where $\alpha > \alpha'$. In other words, we will bend \mathcal{L} and \mathcal{L}' , but bend the latter more than the first. The Floer differential will be as in the ordinary Fukaya category.*

In order to define composition, we will bend $\mathcal{L}_0, \dots, \mathcal{L}_k$, and chose angle $\alpha_0 < \dots < \alpha_k$, and define our products by counting polygons with $k + 1$ sides.

Special Case: If W has non-degenerate isolated quadratic singularities. This is satisfied by a “symplectic” Lefschetz fibration, which generalizes the usual Kahler notion. ($W = \sum z_i^2$)

We have special objects called thimbles, which are half rays starting at the critical points, avoiding all other critical points, and go to infinity. The thimble D_γ associated to such a ray γ is defined as the set of points on $W^{-1}(\gamma)$ whose images converge to the critical point. One can check in the local model that this is a smooth Lagrangian manifold which is topologically a ball, which can be chosen to be admissible.

The intersection of D_γ with each fiber along γ is called the vanishing cycle V_γ .

Consider r critical points and a choice $(\gamma_1, \dots, \gamma_r)$ ordered disjoint non-intersecting paths with endpoints at each of the r critical points, and horizontal at infinity.

Given our earlier convention on morphisms, we see that the space of morphisms

$$CF^*(D_i, D_j) = 0$$

if $i > j$,

$$CF^*(D_i, D_j) = \mathbb{C} \cdot \mathcal{I}d$$

if $i = j$, and otherwise

$$CF^*(D_i, D_j) = CF^*(V_i, V_j)$$

since all intersections. Indeed, the intersections between D_i and D_j all occur over a single intersection point of γ_i and γ_j . Further, the maximum principle (together with a careful choice of the almost complex structure), guaranteeing that all holomorphic discs are contained in a fiber.

Theorem 0.3. *(Seidel) The Lagrangians $(D_{\gamma_1}, \dots, D_{\gamma_r})$ form an exceptional collection, and generate $D^b(\mathcal{F}(Y, W))$. Changing the curves γ_i changes the exceptional collection by a mutation.*

Corollary 0.4. *$D^b(\mathcal{F}(Y, W))$ is determined by a finite amount of data.*

Heuristic: If W were holomorphic, then $\text{Re}(W)$ is a real Morse function. The gradient floer of this real part is a Hamiltonian vector field on each Lagrangian. But, if we flow for a long time, any Lagrangian gets “caught” at the critical points of W .