Phases of $\mathcal{N}=2$ theories in 1+1 dimensions with boundaries

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2d QFT's with (2,2) supersymmetry

moduli space $= \mathcal{M}_K \times \mathcal{M}_C$

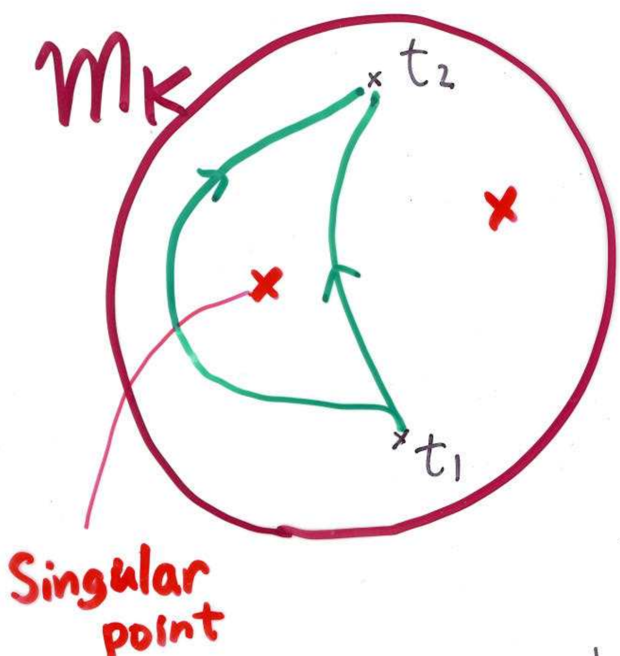
Kähler Complex

D-branes preservind $\mathcal{N}=2$ SUSY
- A-branes
- B-branes

$\mathcal{C}_B$: Category of B-branes

~ **Chiral ring** of "the set" of all B-branes

- depends holomorphically on $c \in \mathcal{M}_C$
- invariant under deformation of $t \in \mathcal{M}_K$



$$\begin{array}{c} \mathcal{C}_B(t_2, c) \\ \uparrow \text{ equivalence of Category} \\ \mathcal{C}_B(t_1, c) \end{array}$$

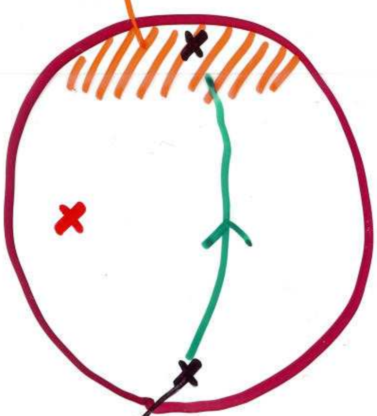
The equivalence depends on the homotopy class of the path

Examples

① $G(x_1,\dots,x_N)$ degree N polynomial

σ-model on $X_G = \{G=0\} \subset \mathbb{P}^{N-1}$

$\mathcal{M}_K$



Landau Ginzburg orbifold

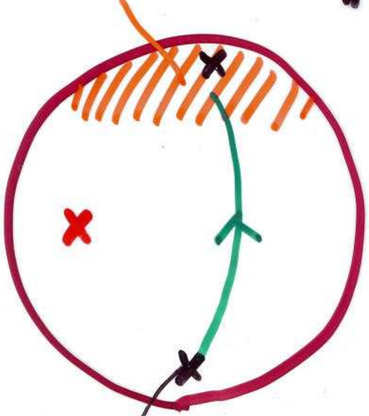
$W = G(x)/\mathbb{Z}_N$

$$\begin{tikzcd} D^b(\mathrm{Coh}\, X_G) \\ MF_{\mathbb{Z}_N}(G) \arrow[u, "\approx"', "\text{Orlov 2005}"] \end{tikzcd}$$

"matrix factorizations"

②

σ-model on $\mathcal{O}_{\mathbb{P}^{N-1}}(-N)$



Orbifold $\mathbb{C}^N/\mathbb{Z}_N$

$$\begin{tikzcd} D^b(\mathrm{Coh}\, \mathcal{O}_{\mathbb{P}^{N-1}}(-N)) \\ D^b_{\mathbb{Z}_N}(\mathbb{C}^N) \arrow[u, "\approx"', "\text{McKay correspondence}"] \end{tikzcd}$$

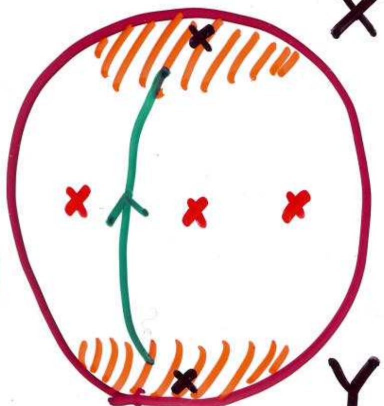
③ Hypersurfaces ($\overset{\text{or}}{\text{Compl. int.}}$) in **Toric Variety**

.... **multidimensional $\mathcal{M}_K$**

④ Two CY's which are NOT birationally equivalent.

Rødland

KH-Tong



$X \subset G(2,7)$

Y = Pfaffian CY

$$D^b(\mathrm{Coh}\,X) \underset{\text{Borisov-Caldararu}}{\overset{\text{Kuznetsov}}{\approx}} D^b(\mathrm{Coh}\,Y)$$

⑤ Non-CY cases

e.g. $G(x_1, \cdots, x_N)$ degree $d \neq N$ $\begin{matrix} d < N \\ d > N \end{matrix}$

IR	UV	×	σ-model on X_G + (d−N) vacua	...	$D^b(\mathrm{Coh}\,X)$ + (d−N) objects
↑ RG	RG ↓				$\approx$ ↕ Orlov
UV	IR	×	LG orbifold $G/\mathbb{Z}_d$ + (N−d) vacua	...	$MF_{\mathbb{Z}_d}(G)$ + (N−d) objects

We'd like to understand this. i.e.

We want to put this into the standard physics framework:

$\boxed{\text{2D Q.F.T. WITH BOUNDARY}}$

★ Then, Applications, e.g., to

★ **Stability** — RG flow analysis

★ **Homological Mirror Symmetry** — dualization.

★ Also, the way to understand category equivalence itself may be mathematically new or interesting....

Linear Sigma Models (LSMs)

Witten 1979, 1993

= a family of 2d (2,2) QFTs / $\mathcal{M}_K \times \mathcal{M}_C$

that reduce to σ-model / orbifold / LG-model

in the respective domain of $\mathcal{M}_K$

Data $(\mathbf{T}, \mathbf{\Phi}, \mathbf{t}, \mathbf{W})$ $\quad t \in \mathcal{M}_K \quad W \in \mathcal{M}_C$

T = a compact abelian group "Gauge group"

Φ = a representation$/\mathbb{C}$ of T ⇒ "matter fields"

$t = r - i\theta \qquad r \in \mathfrak{t}^*$ "Fayet-Iliopoulos parameter"

$\theta \in \mathfrak{t}^*/2\pi Q_T$ "Theta angle"

$[\, \mathfrak{t} = Lie(T),\ Q_T \subset \mathfrak{t}^*$ weight lattice $]$

W = a T-invariant polynomial function of Φ

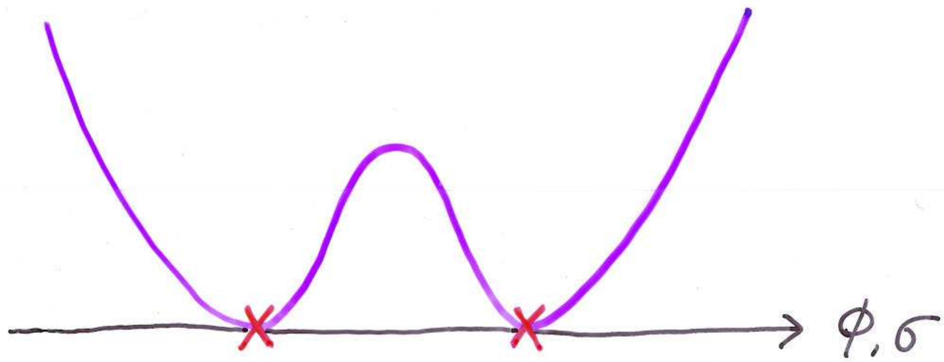
"Superpotential"

$\boldsymbol{\mu} : \Phi \longrightarrow \mathfrak{t}^*$ moment map $\qquad \mu(0) = 0$

Choose a T-inv norm in Φ, Φ^* $\quad |\cdots|^2$

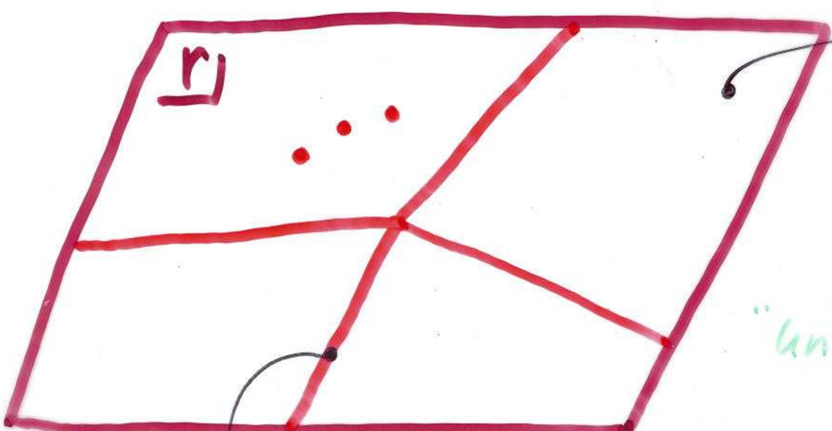
Scalar fields : ϕ in Φ & σ in $\mathfrak{t}_{\mathbb{C}}$

potential $U(\phi,\sigma) = |\sigma\cdot\phi|^2 + \frac{e^2}{2}|\mu(\phi)-r|^2 + |dW(\phi)|^2$



At low energies, the theory reduces to the BOTTOM of the potential, $U(\phi,\sigma)=0$

$\Leftrightarrow \quad \sigma\cdot\phi = 0, \quad \mu(\phi)=r, \quad dW(\phi)=0$



generic r : $\mu(\phi)=r \Rightarrow$

$\mathrm{Stab}(\phi)=\{1\}$ or discrete

$\therefore \sigma\cdot\phi=0 \Rightarrow \sigma=0$

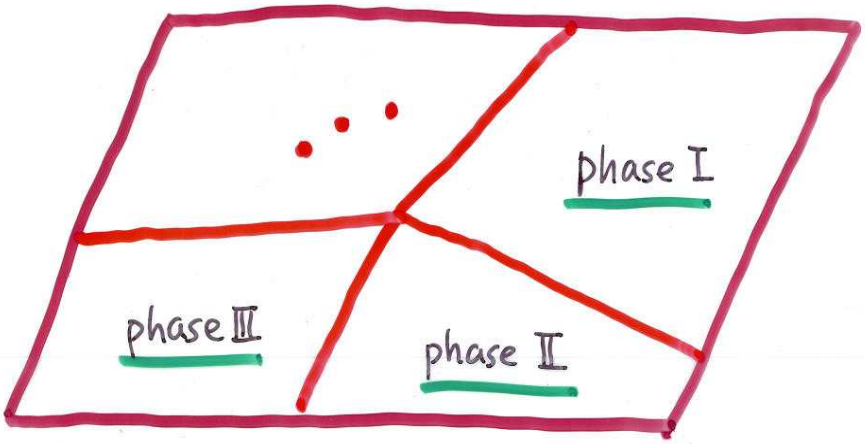
"unbroken gauge group"

Special r: $\exists\phi_0$, $\mu(\phi_0)=r$, $\mathrm{Stab}(\phi_0)\cong U(1)$ or larger

Indefinitely large σ allowed

Singularity !

The space $\{r\}$ $=\mathfrak{t}^*$ is separated into **PHASES**



$r \in$ interior of a phase i $\Rightarrow$ BOTTOM $U=0$

is $\mathbf{X_r} = \{(\phi, \sigma=0) \mid \mu(\phi)=r,\ dW(\phi)=0\}/T$

$$= (\mu^{-1}(r) \cap \mathrm{Crit}\,W)/T \cong ((\Phi - \underbrace{\Delta_i}_{\text{Bad Orbits}}) \cap \mathrm{Crit}\,W)/T_{\mathbb{C}}$$

- $W=0 \Rightarrow X_r = \mu^{-1}(r)/T = (\Phi - \Delta_i)/T_{\mathbb{C}}$ toric variety
- $W \neq 0 \Rightarrow$
  - X_r = hypersuface / complete intersection in toric variety

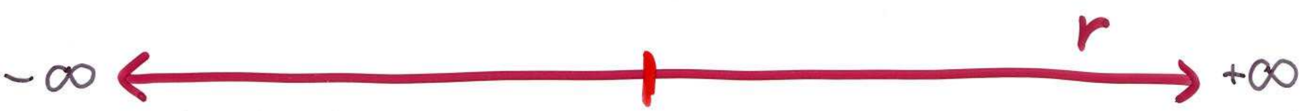
    or
  - X_r = 1 point $[\phi_*]$, but motion in some $\Phi' \subset \Phi$ (affine plane through ϕ_*) is massless

    LG model $W|_{\Phi'}$ **mod.** $\mathrm{Stab}(\phi_*) \subset T$

    or
  - mixture.

Example $T = U(1)$, $\Phi = \mathbb{C}^{N+1} \ni (p, x_1, \dots, x_N) \xmapsto{g \in T = U(1) \subset \mathbb{C}^\times = T_{\mathbb{C}}} (g^{-d}p, g x_1, \dots, g x_N)$

$$W = p\,\underbrace{G(x_1, \dots, x_N)}_{\text{degree } d \text{ polynomial}}$$



<u>$r > 0$ phase</u> $\Delta_+ = \{ x_1 = x_2 = \cdots = x_N = 0 \}$

$$X_+ = \left\{ (p, x) \,\middle|\, \begin{matrix} x \neq 0 \\ G(x) = 0, \ p \frac{\partial G}{\partial x_i}(x) = 0 \end{matrix} \right\} / \mathbb{C}^\times$$

$$\overset{\text{generic } G}{=} \{ x \neq 0 \mid G(x) = 0 \} / \mathbb{C}^\times$$

$$= X_G \subset \mathbb{CP}^{N-1} \quad \text{degree } d \text{ hypersurface}$$

<u>$r < 0$ phase</u> $\Delta_- = \{ p = 0 \}$

$$X_- = \{ p = \langle p \rangle \neq 0, \ x = 0 \} / \mathbb{C}^\times \quad (1 \text{ point})$$

- <u>$\mathrm{Stab} = \mathbb{Z}_d$</u>
- $x_1, \dots, x_N$ massless

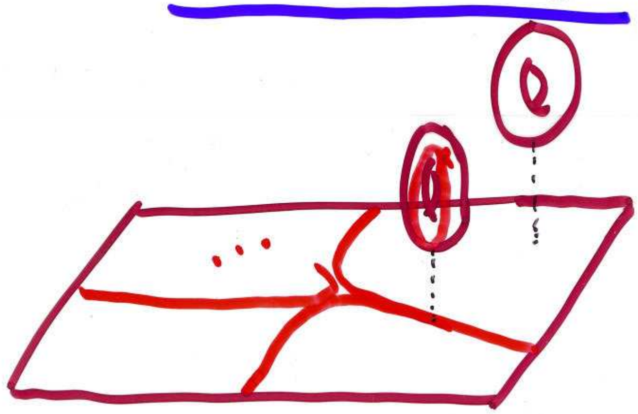
$$W = \langle p \rangle G(x_1, \dots, x_N) / \mathbb{Z}_d \; : \; x_i \to \omega x_i \quad \omega^d = 1$$

LG orbifold

Assume CY condition $T \subset SL(\Phi)$ $\quad \det_{\Phi} g = 1 \quad \forall g \in T$

The QUANTUM theory depends on r but also on

$$\text{THETA ANGLE} \quad \theta \in \mathfrak{t}^{*}/2\pi Q_T \cong T^{*}$$

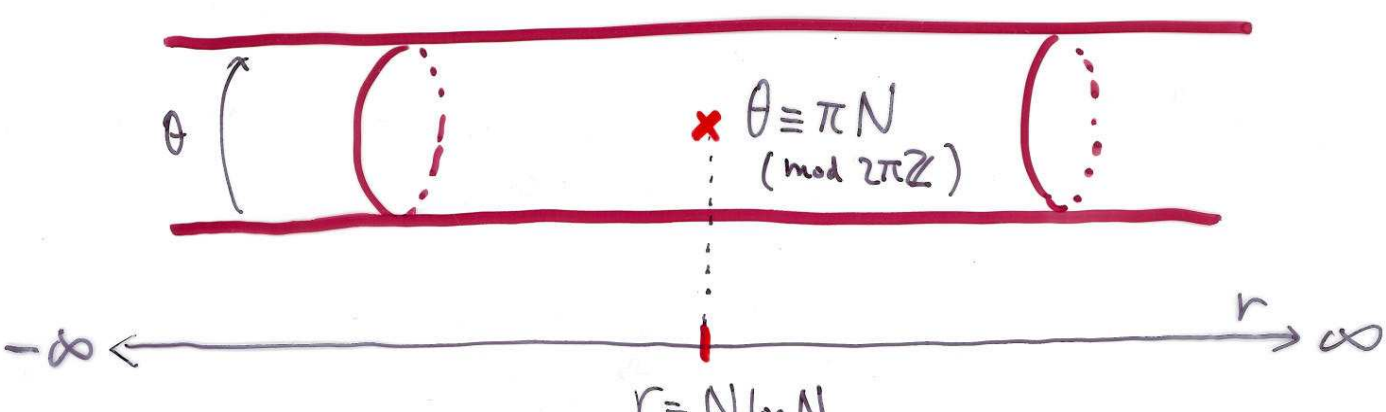


Singularity = a hypersurface $\subset \{(r,\theta)\} \cong T^{*}_{\mathbb{C}}$

asymptotes to classical sing $\subset \{r\}$

$$\boxed{\mathcal{M}_K = T^{*}_{\mathbb{C}} \setminus \overset{\text{(quantum)}}{\text{sing.}}}$$

The Example (CY condition : $d = N$)



In what follows, assume

- CY condition $T \subset SL(\Phi)$ [axial R-symmetry]

- Homogeneous superpotential [Vector R-symmetry]

  i.e. $\exists\, U(1)$ action on Φ, $\phi \mapsto R_\lambda(\phi)$

  s.t. $$W(R_\lambda(\phi)) = \lambda^2 W(\phi)$$

  also $R_{-1}(\phi) \cong \phi$ [integrality of Vector R-charge]

Example with $d=N$: Both O.K.

$$R_\lambda(p, x_1, \cdots, x_N) = (\lambda^2 p, x_1, \cdots, x_N)$$

$$G = p\,G(x_1, \cdots, x_N)$$

D-branes in LSM

— Gauge invariant & homogeneous matrix factorizations of W

(V, Q, ρ, R)

$V = V^{ev} \oplus V^{od}$ $\mathbb{Z}_2$ graded vector space

$Q \in \mathbb{C}[\phi] \otimes \mathrm{End}^{od}(V)$, $Q^2 = W \cdot \mathrm{id}_V$

$\rho : T \to GL^{ev}(V)$ representation

$$\rho(g)^{-1} Q(g \cdot \phi)\, \rho(g) = Q(\phi)$$

$R : U(1) \to GL^{ev}(V)$ representation

$$R(\lambda)\, Q(R_\lambda(\phi))\, R(\lambda)^{-1} = \lambda\, Q(\phi)$$

compatibility: $\rho(g) R(\lambda) = R(\lambda) \rho(g)$

integrality: $R(-1) = \sigma_V = \begin{cases} 1 & \text{on } V^{ev} \\ -1 & \text{on } V^{od} \end{cases}$

Explicitly, with a choice of basis $\left[\, T = \overbrace{U(1)\times\cdots\times U(1)}^{k} \,\right]$

$$Q(\phi) = \begin{pmatrix} 0 & A(\phi) \\ B(\phi) & 0 \end{pmatrix}, \quad A(\phi)B(\phi) = B(\phi)A(\phi) = W(\phi)\mathbb{1}_r$$

$$\rho(g) = \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_{2r}} \end{pmatrix} \qquad \begin{array}{l} q_i = (q_i^1, \cdots, q_i^k) \in Q_T \cong \mathbb{Z}^k \\ g = (g_1, \cdots, g_k) \mapsto g^{q_i} = g_1^{q_i^1} \cdots\cdots g_k^{q_i^k} \end{array}$$

$$R(\lambda) = \begin{pmatrix} \lambda^{R_1} & & \\ & \ddots & \\ & & \lambda^{R_{2r}} \end{pmatrix} \qquad R(-1) = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & -\mathbb{1}_r \end{pmatrix}$$

$$\rho(g)^{-1} Q(g\cdot\phi)\, \rho(g) = Q(\phi)$$

$$R(\lambda)\, Q(R_\lambda(\phi))\, R(\lambda)^{-1} = \lambda\, Q(\phi)$$

$\rightsquigarrow$ Brane-Antibrane system with tachyon condensation

$$\text{brane} = \bigoplus_{i=1}^{r} \mathcal{W}(q_i) \underset{A+B^\dagger}{\overset{B+A^\dagger}{\rightleftarrows}} \text{antibrane} = \bigoplus_{j=r+1}^{2r} \mathcal{W}(q_j)$$

$q \in Q_T$ $\quad \mathcal{W}(q)$: "Wilson line brane"

$\longleftrightarrow$ $\rho(g) = g^q$ component

$\sim$ free graded $\mathbb{C}[\phi]$ module shifted by q

$(V, Q, \rho, R) \longrightarrow$ worldsheet boundary interaction $P\exp\left(-i\int_{\partial\Sigma} \mathcal{A}_t\, dt\right)$

$$\boxed{\mathcal{A}_t = \rho_*(v_0 - \mathrm{Re}(\sigma)) + \frac{1}{2}\{Q, Q^\dagger\} - \frac{1}{2}\sum_{i=1}^{N}\left(\psi^i \partial_i Q + \mathrm{c.c.}\right)}$$

Gauge transformation

$$i\mathcal{A}_t \to \rho(g)\, i\mathcal{A}_t\, \rho(g)^{-1} + \rho(g)\,\partial_t\,\rho(g)^{-1}$$

R-symmetry

$$i\mathcal{A}_t \to R(\lambda)^{-1}\, i\mathcal{A}_t\, R(\lambda)$$

$\mathcal{N}=2$ supersymmetry

0 if $Q^2 \propto \mathrm{id}_V$

$$\delta\mathcal{A}_t = -\mathrm{Re}\left\{\sum_{i=1}^{N}\bar{\epsilon}\psi^i\partial_i Q^2 - [\bar{\epsilon}Q^\dagger, Q^2]\right\}$$

$$+ i\mathcal{D}_t(\bar{\epsilon}Q + \epsilon Q^\dagger) - i(\dot{\bar{\epsilon}}Q + \dot{\epsilon}Q^\dagger)$$

if $Q^2 = W\mathrm{id}$

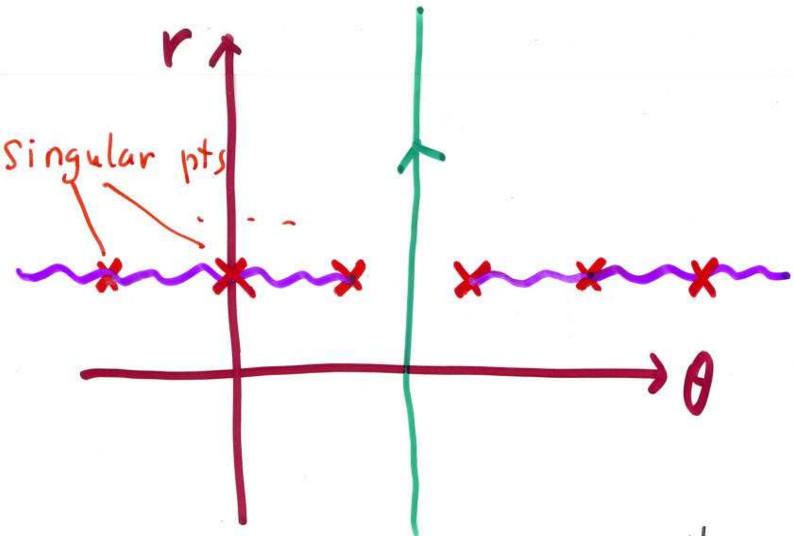
cancels the "Warner term"

$$\delta S_{\mathrm{bulk}} = -\mathrm{Re}\int_{\partial\Sigma} dt \sum_{i=1}^{N}\bar{\epsilon}\psi^i\partial_i W$$

The Grade Restriction Rule "GRR"

$T = U(1)$ If $\partial\Sigma \neq \phi$, $\int_\Sigma \frac{i}{2\pi} F_A \notin \mathbb{Z}$ (for B-brane)

$\Rightarrow$ θ is not a periodic parameter $\theta \not\equiv \theta + 2\pi$



Draw two cuts with a window of length 2π.

We consider paths that go through that window

$$T: \phi = (\phi_1, \dots, \phi_N) \mapsto (g^{Q_1}\phi_1, \dots, g^{Q_N}\phi_N)$$

define $$S := \sum_{Q_i > 0} Q_i \quad \left(= \frac{1}{2}\sum_{i=1}^{N} |Q_i| \quad \text{CY assumed} \right)$$

$$N_{\text{window}} = \left\{ q \in \mathbb{Z} \;\middle|\; -\frac{S}{2} < \frac{\theta}{2\pi} + q < \frac{S}{2} \quad \text{for any } \theta \text{ in that window} \right\}$$

e.g. quintic

$S = 5$ window $-\pi < \theta < \pi \Rightarrow N = \{0, \pm 1, \pm 2\}$

window $-3\pi < \theta < -\pi \Rightarrow N = \{-1, 0, 1, 2, 3\}$

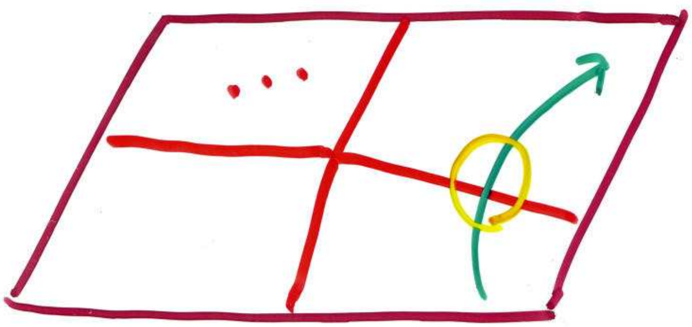
GRR : Along a path that goes through a window

We only admit LSM branes based on

$\mathcal{W}(q)$'s with $q \in \mathbf{N}_{window}$

[i.e. $\rho(g) \cong \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_{2r}} \end{pmatrix} \Rightarrow \forall q_i \in \mathbf{N}_{window}$]

rank $T > 1$: Band restriction rule.



At the phase boundary

$\mathrm{Stab}(\phi_0) \cong U(1)$.

Apply GRR to that $U(1)$.

i.e. Along a path that goes through a window

in the θ-parameter of that $U(1)$,

(V, Q, ρ, R) allowed only if $\rho(g) \cong \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_{2r}} \end{pmatrix}$

$\forall q_i \big|_{\mathrm{Stab}(\phi_0)} \in \mathbf{N}_{window}$

$\mathcal{I}_{\text{window}}$ = Category of <u>Grade Restricted</u> (Band) LSM branes.

For the example, $\left[\begin{array}{l} T = U(1),\ \Phi = \mathbb{C}^{N+1} \ni \underset{-N\ \ 1\ \cdots\ 1}{(p, x_1, \cdots, x_N)} \\ W = pG(x) \end{array} \right]$

We will see

<u>LSM</u>

$$
\begin{tikzcd}
& MF_{\mathbb{C}^\times}(pG(x)) \arrow[dd, phantom, "\cup" description] \arrow[ddd, bend right] \arrow[ddd, bend left] & \\
& & \\
& \mathcal{I}_{\text{window}} \arrow[dl, "\approx"'] \arrow[dr, "\approx"] & \\
MF_{\mathbb{Z}_N}(G(x)) & & D^b(\mathrm{Coh}\, X_G)
\end{tikzcd}
$$

<u>LG orbifold</u> **<u>CY σ-model</u>**

LSM → LG   "set $p=1$"

(V, Q, ρ, R) a LSM brane $\left[Q(p,x)^2 = pG(x)\, \mathrm{id}_V, \cdots, \cdots \right]$

Set $\bar{Q}(x) = Q(1,x)$, $\bar{Q}(x)^2 = G(x)\, \mathrm{id}_V$ ✓ m.f. of $G(x)$

$$\omega^N = 1 \Rightarrow \rho(\omega)^{-1} \bar{Q}(\omega x) \rho(\omega) = \rho(\omega)^{-1} Q(\underset{\omega^{-N} 1}{\overset{\shortparallel}{1}}, \omega x) \rho(\omega) = Q(1,x) = \bar{Q}(x)$$

✓ $\mathbb{Z}_N$-invariance

(Set $\bar{\rho}(\omega) = \rho(\omega)$.)

$$R(\lambda) \underbrace{Q(\lambda^2 \cdot 1, x)}_{\shortparallel} R(\lambda)^{-1} = \lambda Q(1,x)$$

$$\rho(\lambda^{2/N})^{-1} Q(1, \lambda^{2/N} x) \rho(\lambda^{2/N}) \qquad \boxed{\therefore \bar{R}(\lambda) = R(\lambda) \rho(\lambda^{2/N})^{-1}} \Rightarrow$$

$$\bar{R}(\lambda) \bar{Q}(\lambda^{2/N} x) \bar{R}(\lambda)^{-1} = \lambda \bar{Q}(x)$$ ✓ R-symmetry

Note $\bar{R}(e^{\pi i}) \bar{\rho}(e^{2\pi i/N}) = R(-1) = \sigma_V$ ···· required integrality in LG orbifold

Thus $(V, \bar{Q}, \bar{\rho}, \bar{R})$ is a brane of the LG orbifold $W = G(x)/\mathbb{Z}_N$

LG → LSM with GRR [Note: $S=N$]

$(V, \overline{Q}, \overline{\rho}, \overline{R})$ a brane of LGO

$$\left[\begin{array}{l} \overline{R}(\lambda) = \begin{pmatrix} \lambda^{\overline{R}_1} & & \\ & \ddots & \\ & & \lambda^{\overline{R}_{2r}} \end{pmatrix} \\ \overline{R}(e^{\pi i})\overline{\rho}(e^{2\pi i/N}) = \begin{pmatrix} \mathbb{1}_r & \\ & -\mathbb{1}_r \end{pmatrix} \end{array}\right]$$

$$\left.\begin{array}{ll} \exists.1 \ R_i \in 2\mathbb{Z} & i=1\cdots r \\ \qquad\quad 2\mathbb{Z}+1 & i=r+1,\cdots,2r \\ \exists.1 \ q_i \in \mathbf{N}_{\text{window}} & i=1\cdots 2r \end{array}\right\} \text{s.t.} \quad \boxed{\overline{R}_i = R_i - \frac{2q_i}{N}}$$

define $\boldsymbol{\rho}(g) := \begin{pmatrix} g^{q_1} & & \\ & \ddots & \\ & & g^{q_{2r}} \end{pmatrix}$, $\mathbf{R}(\lambda) := \begin{pmatrix} \lambda^{R_1} & & \\ & \ddots & \\ & & \lambda^{R_{2r}} \end{pmatrix}$

Then $\overline{R}(\lambda) = R(\lambda)\,\rho(\lambda^{2/N})^{-1}$

In particular, $\overline{\rho}(e^{2\pi i/N}) = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \overline{R}(e^{\pi i})^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{1}{\underset{\shortparallel}{R(e^{\pi i})}} \rho(e^{2\pi i/N})$

Orbifold invariance $\Rightarrow$ $\rho(e^{2\pi i/N})^{-1}\,\overline{Q}(e^{2\pi i/N}x)\,\rho(e^{2\pi i/N}) = \overline{Q}(x)$

$\therefore$ $\rho(z)^{-1}\,\overline{Q}(zx)\,\rho(z)$ is a function of z^N (& x).

$\overline{Q}(zx)$ ... polynomial in z

$\rho(z)^{-1}(\cdots)\rho(z)$ can change powers of z at most by $z^{\pm(N-1)}$

$\therefore$ polynomial in z^N (no negative power)

$$\rho(z)^{-1}\overline{Q}(zx)\rho(z) = \overline{Q}_0(x) + z^N\overline{Q}_1(x) + z^{2N}\overline{Q}_2(x) + \cdots$$

Now put

$$\mathbf{Q}(p,x) := \overline{Q}_0(x) + p\,\overline{Q}_1(x) + p^2\,\overline{Q}_2(x) + \cdots$$

Then $\rho(g)^{-1}Q(g^{-N}p, gx)\rho(g) = \overline{Q}_0(x) + (g^{-N}p)\cdot g^N\overline{Q}_1(x) + \cdots$

$$= Q(p,x)$$

✓ Gauge invariance

$$R(\lambda)Q(\lambda^2 p, x)R(\lambda)^{-1} = \overline{R}(\lambda)\underbrace{\rho(\lambda^{2/N})Q(\lambda^2 p, x)\rho(\lambda^{2/N})^{-1}}_{\begin{array}{c}\parallel\\ Q(p,\lambda^{2/N}x)\end{array}}\overline{R}(\lambda)^{-1}$$

$$= \lambda Q(p,x)$$

✓ R-symmetry

$$\overline{Q}(x)^2 = G(x)\,\mathrm{id}_V \quad \rightsquigarrow \quad Q(p,x)^2 = p\,G(x)\,\mathrm{id}_V$$

✓ mat. fac. of $pG(X)$

Thus, we obtain a LSM brane (V, Q, ρ, R)

grade restricted

LSM → σ-model

At $r \gg 0$, P & transverse to $G(X)=0$ are heavy

→ integrate them out!

But p is in $Q(p, X)$ .... boundary interaction.

Only bulk modes of P are integrated out.

⇝ effective theory including $P|_{\partial\Sigma}$.

We find that we have a 1st order system

$$\boxed{L_{eff} = \int_{\partial\Sigma} i\, p \overleftrightarrow{D}_t \bar{p}\, dt \quad + \cdots}$$

⇒ $[\bar{p}, p] = 1$ $\qquad$ p creation, $\bar{p}$ annihilation

represented on ∞-dim Fock space

	$\vert 0\rangle$,	$p\vert 0\rangle$,	$p^2\vert 0\rangle$,	$\cdots$	$p^k\vert 0\rangle$,	$\cdots$
gauge charge	0	N	$2N$	$\cdots$	kN	$\cdots$
R-charge	0	2	4	$\cdots$	$2k$	$\cdots$

Total CP space $\mathbb{C}^{2r} \otimes \bigoplus_{k=0}^{\infty} p^k |0\rangle = \bigoplus_{i=1}^{2r} \bigoplus_{k=0}^{\infty} \underbrace{\mathbb{C}_{(i)} \otimes p^k |0\rangle}$

gauge charge $q_i + Nk$

R-charge $R_i + 2k$

line b'dle $\mathcal{O}(q_i + Nk)$

Collect those with R-charge $= j$:

$$\mathcal{E}^j := \bigoplus_{\substack{(i,k) \\ j = R_i + 2k}} \mathcal{O}(q_i + Nk)$$

$j_* := \min_i \{R_i\}$

$$R(\mu)\, Q(\mu^2 p, x) = \mu\, Q(p, x)\, R(\mu) \quad \Rightarrow$$

$$\boxed{\cdots 0 \to 0 \to \mathcal{E}^{j_*} \xrightarrow{Q} \mathcal{E}^{j_*+1} \to \cdots \xrightarrow{Q} \mathcal{E}^{j} \xrightarrow{Q} \mathcal{E}^{j+1} \xrightarrow{Q} \cdots} \quad (\star)$$

$Q^2 : \mathcal{E}^j \xrightarrow{pG(x)\times} \mathcal{E}^{j+2}$ is zero over X_G.

$\therefore$ $(\star)$ is a complex of vector bundles.

- ∞-length toward right
- but $\mathcal{E}^j \xrightarrow{Q} \mathcal{E}^{j+1} \xrightarrow{Q} \mathcal{E}^{j+2}$ exact at $j \gg 0$.

$(\star) \overset{\text{qis}}{\approx}$ **finite length complex.**

Example $N=5$ $G(X) = X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5$

At LG : $\bar{Q} = \sum_{i=1}^{5} \Big(\underset{\frac{2}{5}}{X_i}\, \underset{\frac{3}{5}}{\eta_i} + X_i^4 \underset{-\frac{3}{5}}{\bar{\eta}_i} \Big)$ $\{\eta_i, \bar{\eta}_j\} = \delta_{ij}$

$\{\eta_i, \eta_j\} = \{\bar{\eta}_i, \bar{\eta}_j\} = 0$

represented on $\underbrace{\overset{(1)}{|0\rangle}\ \overset{(10)}{\bar{\eta}_i\bar{\eta}_j|0\rangle},\ \overset{(5)}{\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k\bar{\eta}_l|0\rangle}}_{\text{even}},\ \underbrace{\overset{(5)}{\bar{\eta}_i|0\rangle},\ \overset{(10)}{\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k|0\rangle},\ \overset{(1)}{\bar{\eta}_1\cdots\bar{\eta}_5|0\rangle}}_{\text{odd}}$

R-charge : $\tilde{R}_0$ $\quad \tilde{R}_0 - \frac{6}{5}$ $\quad \tilde{R}_0 - \frac{12}{5}$ $\quad \tilde{R}_0 - \frac{3}{5}$ $\quad \tilde{R}_0 - \frac{9}{5}$ $\quad \tilde{R}_0 - \frac{15}{5}$

.... $L=0$ RS branes $(M = \tilde{R}_0 \cdot 5)$

<u>To Large volume</u> : e.g. $\mathcal{N}_{\text{Window}} = \{0, 1, 2, 3, 4\}$

$\boxed{\tilde{R}_0 = 0}$ first solve $\tilde{R}_i = R_i - \frac{2q_i}{5}$ $\left(\begin{array}{l} R_i \in 2\mathbb{Z}/2\mathbb{Z}+1 \\ q_i \in N_\theta \end{array} \right.$

$|0\rangle$ $\quad 0 = 0 - \frac{2\cdot 0}{5} \Rightarrow R=0 \quad q=0$ $\qquad \mathcal{O}(0)$ at $j=0$

$\bar{\eta}_i\bar{\eta}_j|0\rangle$ $\quad -\frac{6}{5} = 0 - \frac{2\cdot 3}{5} \Rightarrow R=0 \quad q=3$ $\qquad \mathcal{O}(3)^{\oplus 10}$ at $j=0$

$\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k\bar{\eta}_l|0\rangle$ $\quad -\frac{12}{5} = -2 - \frac{2\cdot 1}{5} \Rightarrow R=-2 \quad q=1$ $\qquad \mathcal{O}(1)^{\oplus 5}$ at $j=-2$

$\bar{\eta}_i|0\rangle$ $\quad -\frac{3}{5} = 1 - \frac{2\cdot 4}{5} \Rightarrow R=1 \quad q=4$ $\qquad \mathcal{O}(4)^{\oplus 5}$ at $j=1$

$\bar{\eta}_i\bar{\eta}_j\bar{\eta}_k|0\rangle$ $\quad -\frac{9}{5} = -1 - \frac{2\cdot 2}{5} \Rightarrow R=-1 \quad q=2$ $\qquad \mathcal{O}(2)^{\oplus 10}$ at $j=-1$

$\bar{\eta}_1\cdots\bar{\eta}_5|0\rangle$ $\quad -\frac{15}{5} = -3 - \frac{2\cdot 0}{5} \Rightarrow R=-3 \quad q=0$ $\qquad \mathcal{O}(0)$ at $j = \overset{j_{min}}{-3}$

$$
\begin{array}{ccccccccc}
\cdots -4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \\
 & & & & \mathcal{O}(0) & \mathcal{O}(4)^5 & \mathcal{O}(5) & \\
 & & & & \oplus & \oplus & \oplus & \\
 & & & \mathcal{O}(2)^{10} & \mathcal{O}(3)^{10} & \mathcal{O}(7) & \mathcal{O}(8)^{10} & \cdots \\
 & & & \oplus & \oplus & \oplus & \oplus & \\
\cdots 0 \longrightarrow & \mathcal{O}(0) \xrightarrow{x} & \mathcal{O}(1)^5 \xrightarrow{x^4} & \mathcal{O}(5) \xrightarrow{x} & \mathcal{O}(6)^5 \xrightarrow{x^4} & \mathcal{O}(10) \xrightarrow{x} & \mathcal{O}(11)^5 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \mathrm{pr} & \downarrow & \downarrow & \wr\ \text{qis} \\
\cdots 0 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & \boxed{\mathcal{O}(0)} \longrightarrow & 0 \longrightarrow & 0 \longrightarrow & \cdots
\end{array}
$$

$$\boxed{\widetilde{R}_0 = -\frac{2}{5}}$$

$$
\begin{array}{ccccccccc}
\cdots -4 & -3 & -2 & -1 & 0 & 1 & 2 & \cdots \\
 & & & \mathcal{O}(0)^5 \xrightarrow{x} & \mathcal{O}(1) \longrightarrow & \mathcal{O}(5)^5 \longrightarrow & \mathcal{O}(6) \longrightarrow & \\
 & & & \mathcal{O}(3)^{10} \longrightarrow & \mathcal{O}(4)^{10} \longrightarrow & \mathcal{O}(8)^{10} \longrightarrow & \mathcal{O}(9)^{10} \longrightarrow & \\
\cdots 0 \longrightarrow & \mathcal{O}(1) \xrightarrow{x} & \mathcal{O}(2)^5 \xrightarrow{x^4} & \mathcal{O}(6) \xrightarrow{x} & \mathcal{O}(7)^5 \xrightarrow{x^4} & \mathcal{O}(11) \xrightarrow{x} & \mathcal{O}(12)^5 \longrightarrow & \cdots \\
 & & & \downarrow & \downarrow & & & \wr\ \text{qis} \\
\cdots & 0 \longrightarrow & 0 \longrightarrow & \mathcal{O}(0)^5 \xrightarrow{x} & \mathcal{O}(1) \longrightarrow & 0 & \cdots & \\
 & & & \uparrow & & & & \\
\cdots & \longrightarrow & 0 \longrightarrow & \boxed{T^*_{\mathbb{P}^4}(1)} \longrightarrow & 0 \longrightarrow & 0 & \cdot &
\end{array}
$$

$\boxed{\widetilde{R}_0 = -\frac{4}{5}}$

$$\begin{tikzcd}
 & -3 & -2 & -1 & 0 & 1 & \cdots \\
 & & & \mathcal{O}(1)^5 \arrow[r, "x"] \arrow[dr] & \mathcal{O}(2) \arrow[r] & \mathcal{O}(6)^5 \arrow[r] \arrow[dr] & {} \\
 & & \mathcal{O}(0)^{10} \arrow[ur, "x"] \arrow[r] & \mathcal{O}(4)^{10} \arrow[r] \arrow[dr] & \mathcal{O}(5)^{10} \arrow[r] \arrow[ur] & \mathcal{O}(9)^{10} \arrow[r] \arrow[dr] & \cdots \\
0 \arrow[r] & \mathcal{O}(2) \arrow[r, "x"'] \arrow[dd] & \mathcal{O}(3)^5 \arrow[r, "x^4"'] \arrow[ur] \arrow[dd] & \mathcal{O}(7) \arrow[r] \arrow[dd] & \mathcal{O}(8)^5 \arrow[ur] \arrow[dd] & \mathcal{O}(12) \arrow[l] \arrow[r] \arrow[dd] & {} \\
 & & & & & & \wr\ \mathrm{qis} \\
 & 0 \arrow[r] & \mathcal{O}(0)^{10} \arrow[r, "x"] & \mathcal{O}(1)^5 \arrow[r, "x"] & \mathcal{O}(2) \arrow[r] & 0 &
\end{tikzcd}$$

$\wr\ \mathrm{qis}$

$$\cdots \longrightarrow 0 \longrightarrow \boxed{\Lambda^2 T^*_{\mathbb{P}^4}(1)} \longrightarrow 0 \longrightarrow \cdots$$

$\boxed{\widetilde{R}_0 = -\frac{6}{5}}$

$$\begin{tikzcd}
 & -3 & -2 & -1 & 0 & 1 & \\
 & & & \mathcal{O}(2)^5 \arrow[r, "x"] \arrow[dr] & \mathcal{O}(3) \arrow[r] & \mathcal{O}(7)^5 \arrow[r] \arrow[dr] & {} \\
 & \mathcal{O}(0)^{10} \arrow[r, "x"] \arrow[dr] & \mathcal{O}(1)^{10} \arrow[ur, "x"] \arrow[r] & \mathcal{O}(5)^{10} \arrow[r] \arrow[dr] & \mathcal{O}(6)^{10} \arrow[r] \arrow[ur] & \mathcal{O}(10)^{10} \arrow[r] \arrow[dr] & \cdots \\
\cdots\ 0 \arrow[r] & \mathcal{O}(3) \arrow[r, "x"'] & \mathcal{O}(4)^5 \arrow[r] \arrow[ur] & \mathcal{O}(8) \arrow[r] & \mathcal{O}(9)^5 \arrow[ur] & \mathcal{O}(13) \arrow[l] \arrow[r] & {}
\end{tikzcd}$$

$\wr\ \mathrm{qis}$

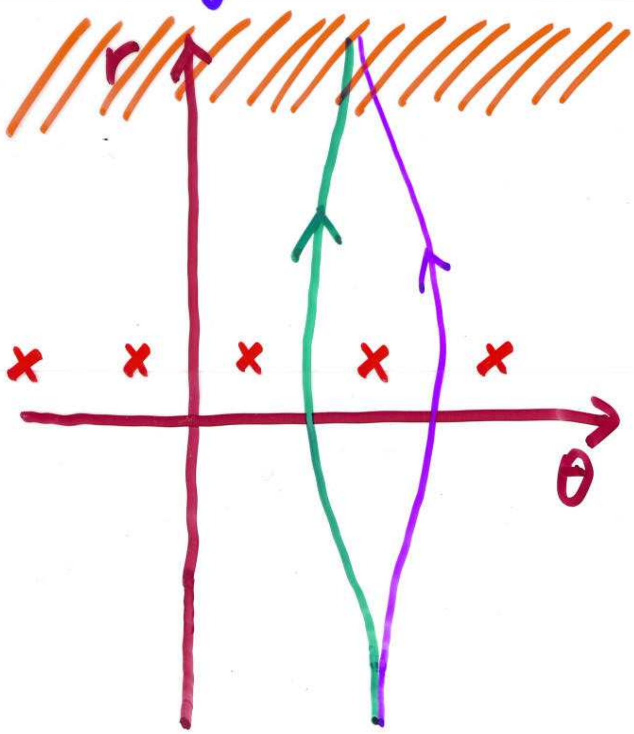
$$\cdots\ 0 \longrightarrow \boxed{\Lambda^3 T^*_{\mathbb{P}^4}(1)} \longrightarrow 0 \ \cdots$$

$\boxed{\widetilde{R}_0 = -\frac{8}{5}}$

$$\begin{tikzcd}
 & -3 & -2 & -1 & 0 & 1 & \\
 & & & \mathcal{O}(3)^5 \arrow[r, "x"] \arrow[dr] & \mathcal{O}(4) \arrow[r, "x^4"] & \mathcal{O}(8)^5 \arrow[r] \arrow[dr] & {} \\
 & \mathcal{O}(1)^{10} \arrow[r, "x"] \arrow[dr] & \mathcal{O}(2)^{10} \arrow[ur, "x"] \arrow[r] & \mathcal{O}(6)^{10} \arrow[r] \arrow[dr] & \mathcal{O}(7)^{10} \arrow[r] \arrow[ur] & \mathcal{O}(11)^{10} \arrow[r] \arrow[dr] & \cdots \\
0 \arrow[r] & \mathcal{O}(0)^5 \arrow[r, "x^4"'] \arrow[ur, "x"] \arrow[dd] & \mathcal{O}(4) \arrow[r] \arrow[dd] & \mathcal{O}(5)^5 \arrow[r] \arrow[ul] \arrow[dd] & \mathcal{O}(9) \arrow[r] \arrow[dd] & \mathcal{O}(10)^5 \arrow[r] \arrow[ul] \arrow[dd] & \mathcal{O}(14) \arrow[r] \arrow[dd] & {} \\
 & & & & & & \wr\ \mathrm{qis} \\
0 \arrow[r] & \mathcal{O}(0)^5 \arrow[r, "x"] & \mathcal{O}(1)^{10} \arrow[r, "x"] & \mathcal{O}(2)^{10} \arrow[r, "x"] & \mathcal{O}(3)^5 \arrow[r, "x"] & \mathcal{O}(4) \arrow[r] & 0 \arrow[r] & {} \\
 & & & & & & \wr\ \mathrm{qis} \\
0 \arrow[r] & \mathcal{O}(-1) \arrow[r] \arrow[uu] & 0 & & & & &
\end{tikzcd}$$

$$\boxed{\mathcal{O}(-1) \cong \Lambda^4 T^*_{\mathbb{P}^4}(1)}$$

Change of window



Different windows

$\leftrightarrow$ Different $\mathcal{I}$'s

$\leftrightarrow$ Different equivalences of $\mathrm{MF}_{\mathbb{Z}_N}(G)$ & $D^b(\mathrm{Coh}\, X_G)$

$$\mathrm{MF}_{\mathbb{C}^\times}(PG(x))$$

$$\cup \qquad \cup$$

$$\begin{tikzcd}
 & \mathcal{I}_1 \arrow[drrr] & \neq & \mathcal{I}_2 \arrow[dr] & \\
\mathrm{MF}_{\mathbb{Z}_N}(G(x)) \arrow[ur] \arrow[urrr] & & & & D^b(\mathrm{Coh}\, X_G)
\end{tikzcd}$$

Difference: <u>**Conifold monodromy**</u>